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SUMMATION OF THE WITTING SERIES IN THE SOLITARY-WAVE PROBLEM

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Some exact solutions of the Euler equations with a free surface in the presence of gravitation forces are found. They are obtained by summing Witting series applied in the theory of solitary waves. It is shown that in some cases, the left-hand half of the constructed flows is close to the left-hand half of the solitary waves.

Introduction. To describe the solitary wave in a fluid of finite depth, Witting [1] proposed a certain power series (the Witting series) and performed its numerical summation. The author showed in [2–4] that the problem of exact summation of this series is reduced to the integration of a special system of ordinary differential equations. The author also generalized the Witting series to the case of periodic waves on water [5–7]. The summation of these generalizations is reduced to the solution of a similar system of equations. The simplest case, where this system consists of three equations, was earlier considered. In the present work, the case where the number of equations is more than three is investigated numerically and analytically.

In the present work, the problem of plane steady-state flows of an ideal incompressible fluid with the free surface in the gravity field is considered. The flows are potential and occur above the even horizontal bottom, and the surface tension is absent. It is assumed that at the left-hand infinity the fluid moves from left to right with horizontal velocity u_0 in a layer of depth h_0. The solitary-wave problem and the flows produced by the Witting series [1] (called Witting flows hereafter) are attributed to this class of problems.

The solitary-wave problem is to determine the free surface in the form of a local rise symmetrically located relative to the vertical axis. This problem depends on one parameter, and either the Froude number ($\mathrm{Fr} = u_0/\sqrt{gh_0} > 1$, where g is the acceleration of gravity) or the Stokes parameter θ ($0 \leqslant \theta < \pi/2$) determined from the equation $\tan\theta/\theta = \mathrm{Fr}^2$ can be used as this parameter. For small amplitudes (or small θ), the solitary-wave problem is investigated analytically. The Witting series is proposed to study the case of amplitudes that are not small. This is the asymptotic expansion in the vicinity of the left-hand infinity. This series was investigated numerically for two cases in [1]: $\theta = \pi/3$ and $\pi/4$. It was shown that based on this series, one can approximately describe solitary waves up to the maximum amplitude. Examples of other types of series for solitary waves can be found in [8, 9].

It was shown in [2–4] that, for $\theta = \pi m/n$, where m and n are integers, the problem of exact summation of the Witting series is equivalent to the solution of a special system of n ordinary differential equations. The Witting series was found to correspond to some other free-boundary flows, rather than solitary waves. For $\theta = \pi/3$, when the system contains three equations (the minimum number), it is easily integrated. The free-surface shape and the streamlines were previously constructed only for this simplest case.

In the present paper, the Witting flows are studied for other values of θ. First, these flows are interesting owing to the fact that they are the exact solutions of hydrodynamics equations on the free surface of which the condition of pressure constancy is satisfied. Secondly, by construction, the left-hand half of the Witting flows should be close to the left-hand half of the solitary waves. It is impossible to guarantee the absence of singular points in the flow and the univalence of the flow; one can guarantee this only on the left-hand side of the flow. Therefore, if one "cuts off" the Witting flow, one can judge, in a study of its left-hand half, only the

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behavior of a solitary wave as the parameter θ is varied. For example, for $\theta = \pi/3$ the Witting flow considered has a point of break at an angle of 120° on the free surface. To the left of this point, the flow is close, with high accuracy, to the solitary wave of maximum amplitude. It is natural to assume that, for $\theta < \pi/3$, we also obtain flows the left-hand half of which is close to the left-hand half of the solitary wave whose amplitude is smaller than the maximum amplitude. This assumption is verified in this study and, apparently, it is true only for $\theta \simeq \pi/3$.

If the Bernoulli integral is used as the boundary condition on a free surface, the initial boundary-value problem is cubically nonlinear. Accordingly, a system of ordinary differential equations which describes Witting flows will be cubically nonlinear. It is, however, known that the wave problem on water is quadratically nonlinear [10]. Below, using Babenko's quadratically nonlinear operator equation, we obtain a quadratically nonlinear system of equations which describes Witting flows and find one integral of this system. Owing to this integral, the system is integrated for $\theta = \pi/4$. When the number of equations in the system is more than four, its solution is found numerically.

1. Formulation of the Problem. We locate the origin of the Cartesian coordinate system at the bottom. The X axis is along the bottom, and the Y axis is directed vertically upward. It is necessary to find $Y = Y_0(X)$, i.e., the free-surface equation which satisfies the condition $\lim_{|X|\to\infty} Y_0(X) = h_0$.

Let Φ and Ψ be the velocity potential and the streamline function, respectively. In the plane of the dimensionless complex potential $\chi = \varphi + i\psi = \theta(\Phi + i\Psi)/h_0 u_0$, the band

$$-\infty < \varphi < \infty \quad (0 < \psi < \theta) \tag{1.1}$$

corresponds to the fluid. The solitary-wave problem will be solved if we find the conformal mapping of this band onto the flow area. We represent this map in the form $Z = X + iY = (h_0/\theta) f(\chi)$, where $f(\chi) = \chi + W(\chi)$. The desired function $W(\chi) = A(\varphi, \psi) + i\,B(\varphi, \psi)$ is defined from the solution of the boundary-value problem

$$\left|\frac{d(W+\chi)}{d\chi}\right|^2 = \frac{1}{1 - 2\nu \operatorname{Im} W}, \quad \nu = \cot\theta \quad (\psi = \theta, \quad \varphi < \varphi_0),$$

$$\operatorname{Im} W = 0 \qquad (\psi = 0, \quad \varphi < \varphi_0), \tag{1.2}$$

$$\lim_{\varphi\to-\infty} \operatorname{Im} W = 0.$$

Here the first equation is the condition of pressure constancy on the free surface (the Bernoulli integral), and the second is the even-bottom condition. The solution of this boundary-value problem is not unique, because the behavior of the solution for $\varphi > \varphi_0$ is not known. The solitary wave is one of the possible solutions; it is obtained if the condition

$$\varphi_0 = +\infty, \qquad \lim_{\varphi\to\infty} \operatorname{Im} W = 0 \tag{1.3}$$

is additionally satisfied.

The search for the solution of (1.2) in the form of a series

$$W = \sum_{j=1}^{\infty} \theta^{2j} W^{(j)}(\chi) \tag{1.4}$$

leads to the shallow-water expansion. This expansion appeared for the first time in another formulation in [11]. All $W_\chi^{(j)}$ are the polynomials of $\cosh^{-2}(\chi/2)$. If we introduce the variable $\zeta = e^\chi$, we shall find that the functions $W^{(j)}$ are the analytical functions of ζ in the vicinity of the point $\zeta = 0$. Therefore, it is natural to reexpand (1.4) and try to search for the solution of the boundary-value problem (1.2) in the form of a power series

$$W = \sum_{j=1}^{\infty} E_j(\theta)\zeta^j, \qquad \operatorname{Im} E_j = 0. \tag{1.5}$$

This is the Witting series. If one substitutes this series into the boundary condition (1.2) and equates the terms at the same degrees e^{φ}, one obtains the following recurrent formulas to find the coefficients E_j:

$$E_2 = E_1^2(-3\nu^2/4 + 1/4),$$
$$E_3 = E_1^3(9\nu^4/16 - 7\nu^2/8 + 1/16),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$E_j(\theta) = E_1^j\,\tilde{E}_j(\nu) \qquad (j \geqslant 2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

The first coefficient remains undetermined. However, its variation leads only to a shift of the solution along the band; therefore, E_1 can be any positive number. The advantage of the Witting series (1.5) over the series (1.4) lies in the fact that $E_j(\theta)$ are found in an easier way compared with $W^{(j)}(\chi)$. Witting [1] found numerically more than 200 terms of the series (1.5).

It is not known whether both series describe the same flow; however, it is clear that these series do not describe a solitary wave. The reason is that the exact solution of problem (1.2), (1.3), i.e., the function $W(\zeta)$, is not analytic for $\zeta = 0$. The paradox is that the series (1.4) and (1.5) were applied to calculate the parameters of solitary waves. This especially concerns the series (1.4), which was used in many studies. The record here belongs to [12], where the expansion is found up to θ^{54}. In this case, the point is approximate calculations, though with high accuracy.

As follows from [2–4], for $\theta = \pi m/n$ the functions

$$P_j(\chi) = W(\zeta\omega^{2j-2}), \quad (\omega = e^{i\theta} \quad j = 1,\dots,n) \tag{1.6}$$

satisfy the system of ordinary differential equations

$$\Big(\frac{dP_{j+1}}{d\chi} + 1\Big)\Big(\frac{dP_j}{d\chi} + 1\Big) = \frac{1}{f_j}, \qquad P_{n+1} \equiv P_1, \qquad j = 1,\dots,n. \tag{1.7}$$

Here $f_j = 1 + i\nu(P_{j+1} - P_j)$. Hence, to solve the boundary-value problem (1.2) in the form of the Witting series (1.5), it is sufficient to integrate system (1.7) and set $W = P_1$.

2. Babenko's Equation. Let an arbitrary analytic function $Q(\chi)$ which satisfies the boundary conditions

$$\operatorname{Im} Q = q(\varphi) \quad (\psi = \theta), \qquad \operatorname{Im} Q = 0 \quad (\psi = 0)$$

be defined in the band (1.1). The operators of restoration of the real part and the normal derivative of the imaginary part on the upper bound of the band are denoted by $\mathbf{H}$ and $\mathbf{N}$, respectively, i.e.,

$$\mathbf{H}q(x) = \operatorname{Re} Q\Big|_{\psi=\theta}, \qquad \mathbf{N}\,q(x) = \frac{\partial}{\partial\psi}\operatorname{Im} Q\Big|_{\psi=\theta}.$$

The Cauchy–Riemann relation

$$\mathbf{H}q_\varphi = \mathbf{N}q \tag{2.1}$$

holds true. With the use of the function $G = 1/(df/d\chi) - 1$ analytic in the band (1.1), the condition of pressure constancy $1/|df/d\chi|^2 = 1 - 2\nu \operatorname{Im} W$ $(\psi = \theta)$ can be written in the form $G = \overline{(df/d\chi)}(1 - 2\nu \operatorname{Im} W) - 1$. Taking into account that $df/d\chi = 1 + \mathbf{N}B + iB_\varphi$, we obtain $G = \mathbf{N}B - 2\nu B(1 + \mathbf{N}B) - i(B - \nu B^2)_\varphi$. The real and imaginary parts of the function G are connected by the operator $\mathbf{H}$. Hence, we have $\mathbf{H}(B - \nu B^2)_\varphi + \mathbf{N}B - 2\nu B(1 + \mathbf{N}B) = 0$. Using the Cauchy–Riemann relations (2.1), we obtain Babenko's equation [10]

$$B + (B - 1/\nu)\mathbf{N}B + \mathbf{N}B^2/2 = 0. \tag{2.2}$$

We reduce this operator equation to a system of ordinary differential equations. To do this, there are two methods. The first and simpler method is based on the analytic continuation of the desired functions beyond the band [4]. The second and more cumbersome method is based on direct summation of the series with the use of generating functions [3]. Only the second method is apparently suitable for Babenko's equation.

Let $\theta = \pi m/n$. We assume that the φ-dependent function which is represented in the form of a power series with respect to e^{φ} belongs to the lth family ($1 \leqslant l \leqslant 2n$) if the power series contains only terms of the form $e^{(l+k\cdot 2n)\varphi}$, where k is an arbitrary nonnegative integer. To sum the series (1.5), it is necessary to introduce $2n$ generating functions

$$B_l(\varphi) = \sum_{j=1}^{\infty} E_{2nj-(2n-l)} e^{\varphi(2nj-2n+l)},$$

each of which is a function of the lth family.

The parameter $B(\varphi, \psi)$ on the upper bound of the band $\psi = \theta$ can be represented as a sum of the functions of $2n$ families:

$$B = \sum_{l=1}^{2n} \sin(l\theta) B_l(\varphi). \tag{2.3}$$

We shall obtain a similar representation for $\mathbf{N}B$ and $\mathbf{N}B^2$. If

$$s_l = a_l e^{l\varphi} + a_{l+2n} e^{(l+2n)\varphi} + \ldots$$

is an arbitrary function of the lth family, we have

$$\mathbf{H}s_l = \frac{\cos(l\theta)}{\sin(l\theta)} s_l, \qquad \mathbf{N}s_l = \frac{\cos(l\theta)}{\sin(l\theta)} s_l', \tag{2.4}$$

where the prime denotes the derivative with respect to φ. This follows from the consideration of the real and imaginary parts of the auxiliary analytic function $S_l = a_l e^{l\chi} + a_{l+2n} e^{(l+2n)\chi} + \ldots$ at the upper bound of the band $\chi = \varphi + i\theta$. Taking into account $\operatorname{Re} S_l = \cos(l\theta) s_l$ and $\operatorname{Im} S_l = \sin(l\theta) s_l$, we obtain the desired formulas (2.4). Applying them in (2.3), we find

$$\mathbf{N}B = \sum_{l=1}^{2n} \cos(l\theta) B_l'(\varphi). \tag{2.5}$$

Taking the square of (2.3), with the use of the formula of summation over the diagonal of all the elements of the square matrix

$$\sum_{i=1}^{2n} \sum_{j=1}^{2n} a_{ij} = \sum_{l=1}^{2n} \Big(\sum_{j=1}^{l-1} a_{j,l-j} + \sum_{j=l}^{2n} a_{j,l+2n-j} \Big)$$

and taking into consideration that the products $B_j B_{l-j}$ and $B_j B_{l-j+2n}$ are functions of the lth family, i.e., formulas (2.4) are suitable for them, we have

$$\mathbf{N}B^2 = \sum_{l=1}^{2n} \frac{\cos(l\theta)}{\sin(l\theta)} \Big[\sum_{j=1}^{l-j} \sin(j\theta)\, \sin(l-j)\theta \cdot (B_j B_{l-j})' + \sum_{j=l}^{2n} \sin(j\theta)\, \sin(l-j)\theta \cdot (B_j B_{l-j+2n})' \Big]. \tag{2.6}$$

Substituting the resulting expressions (2.3), (2.5), and (2.6) into Babenko's equation (2.2) and taking into account that, if the sum of $2n$ functions of various families is zero, we obtain

$$\sin(l\theta) \Big\{ \sum_{j=1}^{l-1} \sin(j\theta) \cos(l-j)\theta \cdot B_j B_{l-j}' + \sum_{j=l}^{2n} \sin(j\theta) \cos(l-j)\theta \cdot B_j B_{l+2n-j} \Big\}$$

$$+ \cos(l\theta) \Big\{ \sum_{j=1}^{l-1} \sin(j\theta)\, \sin(l-j)\theta \cdot B_j B_{l-j}' + \sum_{j=l}^{2n} \sin(j\theta)\, \sin(l-j)\theta \cdot B_j B_{l+2n-j} \Big\}$$

$$+ \sin^2(l\theta) B_l - (1/\nu) \cos(l\theta)\, \sin(l\theta) B_l' = 0. \tag{2.7}$$

Thus, we have found the required quadratically nonlinear system of ordinary differential equations. To simplify it, we apply a discrete Fourier transform. By definition, if an arbitrary vector from $2n$ complex numbers D_l is given, a transform implemented by the formula

$$\hat{D}_j = \sum_{l=1}^{2n} D_l \omega^{(l-1)(j-1)},$$

where $\omega = e^{i\theta} = e^{i\pi m/n}$, is called a discrete Fourier transform. If

$$C_l = \sum_{j=1}^{l} D_j H_{l-j+1} + \sum_{j=l+1}^{2n} D_j H_{l+2n-j+1},$$

then $\hat{C}_p = \hat{D}_p \hat{H}_p$. Owing to this property, Eqs. (2.7) are simplified. After the Fourier transform, the sums in (2.7) disappear.

We introduce new vectors. The left-hand column gives their notation, and the right-hand column the corresponding Fourier transforms:

$$\begin{aligned}
&F_l = \omega^{2l} B_l, && \hat{F}_p = \omega^2 \hat{B}_{p+2},\\
&G_l = \omega^{4l} B_l', && \hat{G}_p = \omega^4 \hat{B}'_{p+4},\\
&C_l = \sum_{j=1}^{l-1} B_j B'_{l-j} + \sum_{j=l}^{2n} B_j B'_{l-j+2n}, && \hat{C}_p = \omega^{p-1} \hat{B}_p \hat{B}'_p,\\
&D_l = \sum_{j=1}^{l-1} F_j B'_{l-j} + \sum_{j=l}^{2n} F_j B'_{l+2n-j}, && \hat{D}_p = \omega^{p-1} \hat{F}_p \hat{B}'_p,\\
&S_l = \sum_{j=1}^{l-1} F_j G_{l-j} + \sum_{j=l}^{2n} F_j G_{l-j+2n}, && \hat{S}_p = \omega^{p-1} \hat{F}_p \hat{G}_p.
\end{aligned} \tag{2.8}$$

Via the new vectors, expression (2.7) can be rewritten in the form

$$(2\omega^{2l} - \omega^{4l} - 1)B_l + \frac{i}{\nu}(\omega^{4l} - 1)B_l' - (\omega^{4l} + 1)C_l + D_l + S_l = 0.$$

Performing the Fourier transform of the last equation, denoting $P_l = \hat{B}_{2l-1}\omega^{2l-2}$, and using the right-hand column of (2.8), after some simplifications we find

$$(P_l' + 1)f_l = (P_{l+2}' + 1)f_{l+1} \qquad (1 \leqslant l \leqslant n). \tag{2.9}$$

Thus, we obtain system (2.9) from Babenko's equation, and each equation in it is a consequence of two adjacent equations (1.7). If system (1.7) can be written in the standard form resolved relative to the derivatives, this resolution is impossible for (2.9), because the product of all the equations in (2.9) gives an identity. Therefore, an integral should exist. This integral is easily found if one sums all the equations in (2.9). With allowance for $\sum_{j=1}^{n} f_j = \text{const}$, we obtain

$$\sum_{j=1}^{n} \frac{dP_j}{d\chi} f_j = \sum_{j=1}^{n} \frac{dP_{j+1}}{d\chi} f_j \quad \text{or} \quad \frac{d}{d\chi}\sum_{j=1}^{n} f_j^2 = 0.$$

Rewriting the last equality, we conclude that system (2.9) should be supplemented by the quadratically nonlinear integral

$$(P_1 - P_2)^2 + (P_2 - P_3)^2 + \ldots + (P_{n-1} - P_n)^2 + (P_n - P_1)^2 = \text{const}. \tag{2.10}$$

For gravitational waves on the surface of a fluid, the Stokes series are found by recurrent formulas with double sums. Longuet-Higgins [13] found a nonlinear transformation which permits one to simplify these formulas such that they contain only unitary sums. These transformations of different series used in wave problems have been performed so far. Apparently, the fact of the square nonlinearity of the initial formulation of the problem is very little known For (1.5), Witting [1] also used recurrent formulas with double sums. The formulas with unitary sums can be derived if one uses one equation of system (1.7) which can be written with allowance for (1.6) in the form

$$\Big(\frac{dW(\chi-2\mathrm{i}\theta)}{d\chi}+1\Big)[1+\mathrm{i}\nu(W(\chi)-W(\chi-2\mathrm{i}\theta))]=\Big(\frac{dW(\chi+2\mathrm{i}\theta)}{d\chi}+1\Big)[1+\mathrm{i}\nu(W(\chi+2\mathrm{i}\theta)-W(\chi))].$$

The initial boundary-value problem (1.2) is equivalent to this quadratically nonlinear, differential-difference equation. Substituting the Witting series (1.5) into it and equating the terms with equal degrees ζ, we obtain the recurrent formula

$$E_j=\frac{\nu}{2\sin(j\theta)\,(j\cos(j\theta)-\nu\sin(j\theta))}\sum_{k=1}^{j-1}kE_kE_{j-k}(\cos(2k\theta)-\cos(2j\theta)).$$

3. System of Four Equations. We shall consider the case $\theta=\pi/4$, which follows the case $\theta=\pi/3$ in complexity. It was partially studied in [3]. Below, we shall construct the streamlines and find explicit formulas for the free-surface shape.

We have the system of four equations

$$\begin{aligned}(P_2'+1)(P_1'+1)&=1/f_1, \qquad (P_3'+1)(P_2'+1)=1/f_2,\\ (P_4'+1)(P_3'+1)&=1/f_3, \qquad (P_1'+1)(P_4'+1)=1/f_4,\end{aligned} \tag{3.1}$$

where $f_1=1+\mathrm{i}(P_2-P_1)$, $f_2=1+\mathrm{i}(P_3-P_2)$, $f_3=1+\mathrm{i}(P_4-P_3)$, and $f_4=1+\mathrm{i}(P_1-P_4)$. We introduce the following new unknowns:

$$\begin{aligned}Q_1&=E_1\zeta+E_3\zeta^3+E_5\zeta^5+E_7\zeta^7+\ldots, \qquad R_1=E_2\zeta^2+E_4\zeta^4+E_6\zeta^6+E_8\zeta^8+\ldots,\\ Q_2&=E_1\zeta-E_3\zeta^3+E_5\zeta^5-E_7\zeta^7+\ldots, \qquad R_2=-E_2\zeta^2+E_4\zeta^4-E_6\zeta^6+E_8\zeta^8-\ldots.\end{aligned} \tag{3.2}$$

As follows from (1.6), they are related to the old unknowns by the relations $P_1=Q_1+R_1$, $P_2=\mathrm{i}Q_2+R_2$, $P_3=-Q_1+R_1$, and $P_4=-\mathrm{i}Q_2+R_2$. System (3.1) is solved using the integral (2.10), which has the form

$$Q_2^2-Q_1^2=(R_2-R_1)^2, \tag{3.3}$$

and also the second explicit integral $f_1f_3=f_2f_4$, which can written in the form

$$R_2-R_1=Q_2Q_1. \tag{3.4}$$

As a result, the order of system (3.1) can be halved. We obtain

$$\frac{dQ_1}{d\chi}=\frac{Q_1(1-Q_1^2)}{\sqrt{(1+Q_1^2)(1-2Q_1^2)}}, \qquad \frac{d(R_1+\chi)}{d\chi}=\sqrt{\frac{(1-Q_1^2)^3}{(1+Q_1^2)(1-2Q_1^2)}}. \tag{3.5}$$

It is necessary to find a solution of this system which satisfies, as follows from (3.2), the conditions

$$\lim_{\varphi\to-\infty}Q_1\mathrm{e}^{-\chi}=E_1, \qquad \lim_{\varphi\to-\infty}R_1\mathrm{e}^{-2\chi}=E_1^2(-3\nu^2/4+1/4).$$

The first equation of the system is solved irrespective of the second equation. After the function $Q_1(\chi)$ is found, the second equation is integrated, and the Witting solution for $\theta=\pi/4$ is given by the formula

$$W=Q_1+R_1. \tag{3.6}$$

At the bottom, i.e., when $\chi=\varphi$, for relatively large negative φ the function Q_1 is real. When Q_1 is increased from 0 to $1/\sqrt{2}$, the parameter φ increases from $-\infty$ to a certain φ^*. As φ further increases, as follows from the first equation of (3.5), the function Q_1 cannot remain real and, hence, the point $\chi=\varphi^*$ is singular. Denoting $u=Q_1^2$ and integrating the first equation of (3.5), we obtain

$$\frac{1}{2}\int_{1/2}^{u}\frac{du}{u(1-u)}\sqrt{(1+u)(1-2u)}=\chi-\varphi^*.$$

This formula is the Christoffel–Schwarz integral, which performs the conformal mapping of the upper half-plane u onto the polygon ADFCB in the plane of the complex potential χ shown in Fig. 1.

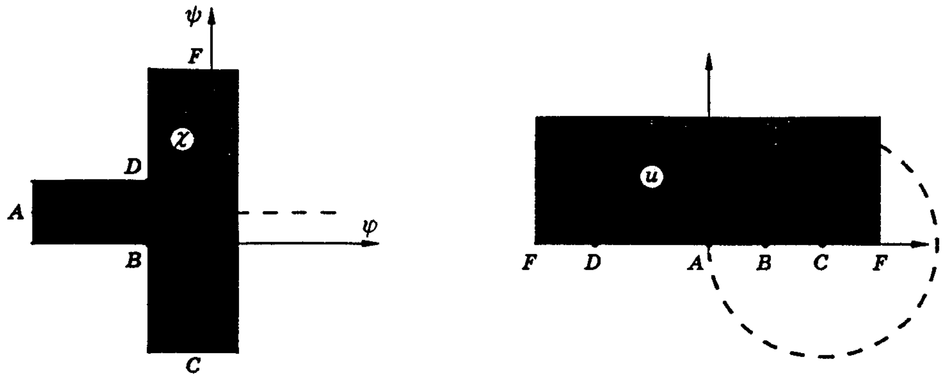

Fig. 1

It is easy to justify that upon this conformal mapping, the upper bound of the band (1.1) corresponds to the circle $|u-1|=1$ or

$$u = 2\cos\alpha e^{\mathrm{i}\alpha} \qquad (-\pi/2 \leqslant \alpha \leqslant \pi/2) \tag{3.7}$$

(shown by the dashed curve) in the plane of the complex potential. Hence, the identity $|1-Q_1^2| = 1$ ($\psi = \pi/4$) holds true on the free surface. Taking into account the consequence of (3.3) and (3.4)

$$(1-Q_1^2)(1+Q_2^2) = 1, \tag{3.8}$$

we find another identity:

$$|1+Q_2^2| = 1 \qquad (\psi = \pi/4). \tag{3.9}$$

To prove two more identities

$$\overline{Q}_1 = -\mathrm{i}Q_2, \qquad \overline{R}_1 = R_2 \qquad (\psi = \pi/4), \tag{3.10}$$

it suffices to substitute $\zeta = r\,\mathrm{e}^{\mathrm{i}\pi/4}$ ($r \geqslant 0$) into the series (3.2). From the first equality in (3.10), we obtain the consequence

$$|1+Q_1^2| = |1-Q_2^2| \qquad (\psi = \pi/4). \tag{3.11}$$

We use these identities to check the condition of pressure constancy on the free surface. It is necessary to prove that the equality

$$\left|\frac{d(W+\chi)}{d\chi}\right|^2 = \frac{1}{1-2\,\mathrm{Im}\,W} \qquad (\psi = \pi/4),$$

which can be written in more detail and with allowance for (3.5) and (3.6) in the form

$$\left|\frac{Q_1(1-Q_1^2)+(1-Q_1^2)^{3/2}}{\sqrt{(1+Q_1^2)(1-2Q_1^2)}}\right|^2 = \frac{1}{1-2\,\mathrm{Im}\,(Q_1+R_1)}, \tag{3.12}$$

is satisfied. We first transform the left-hand side of (3.12). This side will not vary if one divides it by $|1-Q_1^2|^{3/2} = 1$. Using the consequence of (3.8) to transform the numerator $Q_2 = Q_1/\sqrt{1-Q_1^2}$, we find that the left-hand part is equal to $|1+Q_2|^2/|1+Q_1^2||1-2Q_1^2|$. Taking into account the equality $1-2Q_1^2 = (1-Q_2^2)/(1+Q_2^2)$ obtained from (3.8) and also (3.9) and (3.11), we conclude that

$$\left|\frac{Q_1(1-Q_1^2)+(1-Q_1^2)^{3/2}}{\sqrt{(1+Q_1^2)(1-2Q_1^2)}}\right|^2 = \frac{1}{|1-Q_2|^2}. \tag{3.13}$$

Now we transform the right-hand part of (3.12). With allowance for the equalities (3.4) and (3.10), we have

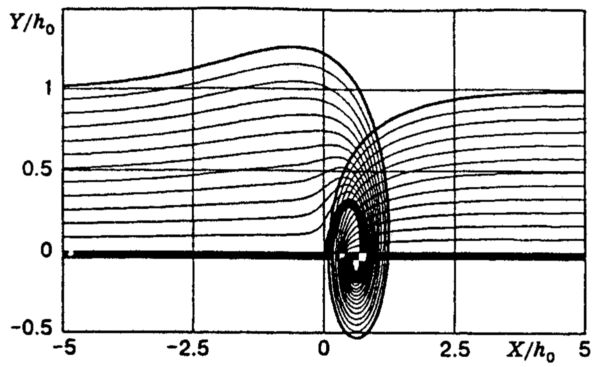

Fig. 2

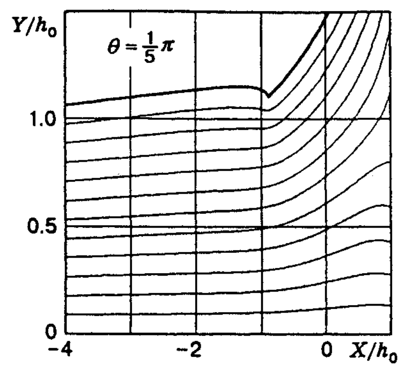

Fig. 3

the chain of equalities to transform the right-hand part of (3.12):

$$\frac{1}{1+\mathrm{i}(Q_1+R_1-\overline{Q}_1-\overline{R}_1)}=\frac{1}{1+\mathrm{i}Q_1-Q_2+\mathrm{i}(R_1-R_2)}$$
$$=\frac{1}{1+\mathrm{i}Q_1-Q_2-\mathrm{i}Q_1Q_2}=\frac{1}{(1-Q_2)(1+\mathrm{i}Q_1)}=\frac{1}{|1-Q_2|^2}.$$

Comparing with (3.13), we conclude that the condition of pressure constancy is satisfied.

In the vicinity of the point $\chi=\varphi^*$, the solution has the form

$$(W+\chi)-(W+\chi)^*=3^{1/3}2^{-5/6}e^{\mathrm{i}\pi/3}(\chi-\varphi^*)^{2/3}.$$

Hence, the fluid boundary is broken with an internal angle of 120°. The even-bottom condition is satisfied for $\varphi<\varphi^*$. At the point $\chi=\varphi^*$, the bottom declines stepwise. With increase in φ, the even-bottom condition is violated up to the point $\varphi^{**}=\varphi^*+\sqrt{2}\pi$. In the vicinity of this point, the solution has the form

$$(W+\chi)-(W+\chi)^{**}=\mathrm{const}\,e^{\mathrm{i}4\pi/3}(\varphi^{**}-\chi)^{4/3}\qquad(\mathrm{const}>0).$$

Hence, a break occurs again; however, for $\varphi>\varphi^{**}$ the bottom is again even and horizontal. This solution can be interpreted as a flow above the uneven site of the bottom positioned for $\varphi^*<\varphi<\varphi^{**}$. The "obstacle length" at the bottom is given by the formula

$$x^{**}-x^*=\frac{4h_0}{\pi}\left\{2\ln(1+\sqrt{2})-\sqrt{2}\right\}.$$

Integrating the consequence of (3.5) $d(R_1+\chi)/du=\sqrt{1-u}/2u$, we arrive at the following statement. If $u(\chi)$ is the conformal mapping of a polygon in the plane of the complex potential χ (Fig. 1) at the upper half-plane u, the Witting solution for $\theta=\pi/4$ is given by the formula

$$\chi+W=\sqrt{u}+\sqrt{1-u}+\frac{1}{2}\ln\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}+\mathrm{const}.$$

Substituting (3.7) into it, we obtain the parametric representation of the free-surface equation

$$X=\frac{4}{\pi}\left(\sqrt{2\,\cos\alpha}\cos\frac{\alpha}{2}+\frac{1}{4}\ln\frac{1-\sin\alpha}{1+\sin\alpha}+\sin\alpha\right)+\mathrm{const},$$
$$Y=\frac{4}{\pi}\left(\sqrt{2\cos\alpha}\,\sin\frac{\alpha}{2}-\cos\alpha+\frac{\pi}{4}\right)\qquad(-\pi/2\leqslant\alpha\leqslant\pi/2).$$

The free-surface shape, which is determined by this formula, and the streamlines are shown in Fig. 2. The two-valence flow above the uneven bottom with a point of self-intersection is found. At the lower point, the fluid is below the rigid wall. The separation of the fluid does not occur because of the negative pressure inside the fluid.

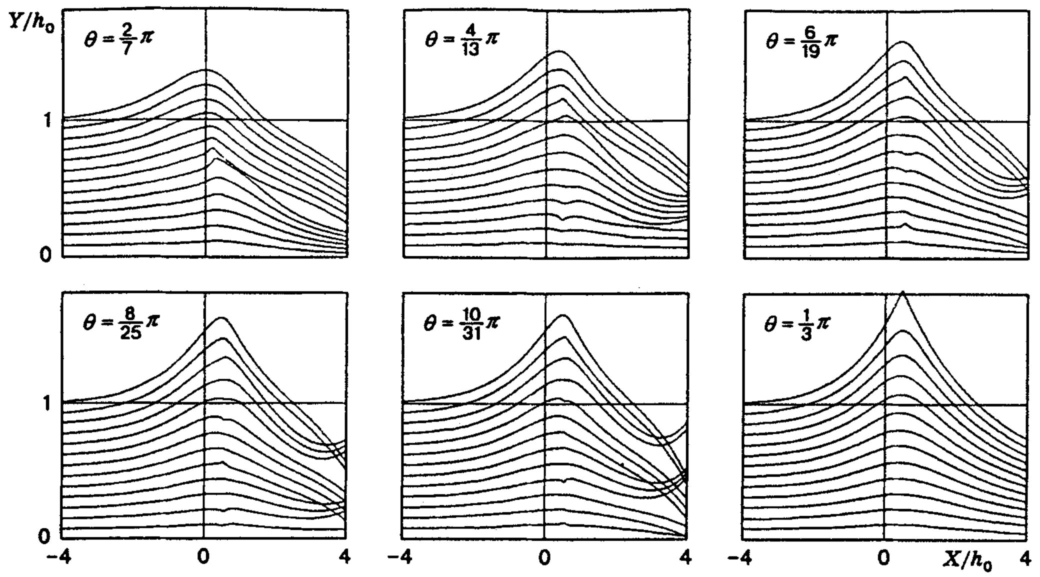

Fig. 4

A flow that is not similar to a solitary wave was obtained. However, this did not stop Witting from constructing a solitary wave for $\theta = \pi/4$ on the basis of this flow, which was found by him numerically. He linearized problem (1.2) using this solution; the perturbed solution was selected so that the symmetry condition relative to the vertical axis was satisfied. As a result, a wave profile close to the results obtained by other authors was found. Now the exact solution can be linearized similarly.

4. Numerical Integration of the System. The case which follows the case $\theta = \pi/4$ in complexity is $\theta = \pi/5$. Here system (1.7) includes five equations. We failed to integrate it exactly. If one denotes $y_j = 1/(dP_j/d\chi+1)(dP_{j+1}/d\chi+1)$ and introduces the independent variable $\zeta = i\nu \int d\chi/\sqrt{f_1 f_2 f_3 f_4 f_5}$, for $\theta = \pi/5$, one can write (1.7) in the following symmetrical form:

$$\frac{dy_1}{d\zeta} = y_3 y_5 - y_2 y_4, \qquad \frac{dy_2}{d\zeta} = y_4 y_1 - y_3 y_5, \qquad \frac{dy_3}{d\zeta} = y_5 y_2 - y_4 y_1,$$
$$\frac{dy_4}{d\zeta} = y_1 y_3 - y_5 y_2, \qquad \frac{dy_5}{d\zeta} = y_2 y_4 - y_1 y_3. \tag{4.1}$$

It is easy to show that this system has the integrals

$$\frac{d}{d\zeta}(y_1 + y_2 + y_3 + y_4 + y_5) = 0, \qquad \frac{d}{d\zeta}(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) = 0;$$

we note that the second integral corresponds to (2.10). It is not known whether the system has other polynomial integrals.

The streamlines obtained by numerical integration of system (4.1) are shown in Fig. 3. The upper streamline corresponds to the free surface. The condition of pressure constancy is satisfied from $-\infty$ to the singular point (the point of break in Fig. 3).

System (1.7) was integrated numerically in the case where the number of equations $-\infty$ was more than five. We confined ourselves to odd n. In this case, there is a simple formula which allows us to write system (1.7) in the standard form resolved with respect to the derivatives:

$$\frac{dP_j}{d\chi} + 1 = \prod_{k=1}^{(n-1)/2} f_{[(2k+j-2) \bmod n]+1} \Bigg/ \sqrt{\prod_{k=1}^{n} f_k} \quad (1 \leqslant j \leqslant n). \tag{4.2}$$

In the case of even n, to derive the corresponding formula it is necessary to use the derivative of the integral

$$\prod_{j=1}^{n/2} f_{2j} = \prod_{j=1}^{n/2} f_{2j-1},$$

and this formula is not simple.

We set $E_1 = 1$. To sum the Witting series, one should solve the Cauchy problem $\lim_{\zeta \to 0} P_j/\zeta = e^{2i(j-1)\theta}$ ($1 \leqslant j \leqslant n$) for system (4.2). This problem was solved by numerical integration in the plane ζ. The fourth-order Runge–Kutta method was used. Since the point $\zeta = 0$ is singular for (4.2), the integration was started at a different point $\zeta \neq 0$. The solution at this point was previously found by numerical summation of the Witting series (1.5).

Figure 4 shows the streamlines for six values of θ. The upper streamline corresponds to the free surface. If the streamlines are smooth everywhere for $\theta = \pi/3$ and there is only one singular point (located at the flow boundary), the flow contains many singular points for $\theta < \pi/3$. Most of them are inside the fluid. The flow becomes multifold, i.e., the streamlines intersect. However, the singular points are located only in the right-hand part of the flow. The left-hand half is free from them and can be used to describe approximately solitary waves whose amplitude is smaller than the maximum amplitude.

It is seen that as θ varies, the flow varies chaotically on the right-hand side and little on the left-hand side. Apparently, the solution given by the Witting series (1.5) has a continuous dependence on the parameter θ on the left-hand side.

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